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CHARACTERISATION OF FRACTIONAL CONVOLUTION OPERATOR IN TIME DOMAIN

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Abstract. Although the Fourier Transform has vital role in many branches of science and technology, it comprises some limitations as well. Particularly, its inability to provide local time-frequency information, which is crucial for analyzing non-stationary signals. To address this issue, the fractional Fourier transform has been serving as an alternative tool for past many years. In this paper, a μ- generalised translation of fractional Fourier transform of one dimensional signal is proposed to characterise a fractional convolution operator in time domain. This operator satisfies some properties, like properties of conventional Fourier transform. Through examples, we have shown the novelty of the proposed characterisation.

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1. Introduction

The Fourier Transform (FT) is a potent tool in both theoretical and applied mathematics. It has various applications as analyzing signals and processing data in the transformed domain, the frequency domain. Moreover, other applications including audio signal processing, communication systems, image processing, seismological signal analysis, econometrics, physical sciences, engineering and quantum mechanics. Fourier analysis is used in storage and transmission of digital images. Its inventiveness and power make it a crucial one in many scientific and engineering fields.

The FT has some limitations, particularly, its inability to provide local time-frequency information, which is crucial for analyzing non-stationary signals. To address this, the fractional Fourier transform (FrFT) was introduced [34]. The FrFT extends the FT by introducing a fractional order α. When $\alpha = \frac{\pi}{2}$, the FrFT can be restricted to FT, and when $\alpha = 0$, it reduce to identity operator [7, 8]. The FrFT, with given order α, rotates the time-frequency plane by an angle α. The properties of the FT are special cases of the FrFT [12].

The FrFT has demonstrated extensive applicability across interdisciplinary domains, including computer tomography and cryptography [2, 10]. Unlike traditional

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FT, the FrFT incorporates an additional parameter α, enhancing its utility in processing disorganized signals and pattern recognition [6, 33]. It plays a significant role in solving differential equations [1, 34]. The development of fractional domain filtering reveals potential improvements over conventional spatial and frequency domain operations [11, 28]. Moreover, fractional convolution techniques exhibit promising applications in RADAR systems and digital watermarking [14, 15]. The use of FrFT in wireless communications and mobile computing is a relevant application [32]. FrFT-based filtering has also proven effective in image representation, compression, and applications within optics and communication systems [16, 23]. The continued exploration of FrFT, particularly in filter design, highlights its expanding potential for practical applications [25].

The Heisenberg uncertainty principle [19, 35] is a fundamental aspect of FrFT. Generalized Heisenberg-type uncertainty principles [26], extending traditional FT uncertainty principles, are crucial in various fields due to their association with numerous inequalities. Additionally, several inequalities and generalizations of the Wigner-Ville distribution related to linear canonical transforms [24, 27] have been explored, contributing to various applications. While fractional convolution operator and their associated theorems are valuable [36] for filtering applications, previous definitions often lack consistency with established FT theories [4]. Notably, convolution theorems developed recently [30] offer enhanced simplicity and coherence compared to traditional FT-based approaches.

Inspired by the work of K. K. Sharma et al. [22], present paper estabalish the characterisation of fractional convolution operator and generalised translations for some angles (α). We have discussed a generalised convolution operator in time domain with the help of a generalised translation operator and derived fractional convolution theorem. Moreover, we have discussed some properties of generalised convolution operator. The results are veryfied by some graphical demonstrations. This paper is organised as follows: some background and preliminaries are given in section 2. In section 3, the main results are discussed. Finally, the conclusions are included in the last section.

2. Background and Preliminaries

2.1. **Fourier Transform.** The FT of the signal $s(t) \in L^2(\mathbb{R})$, denoted by $S(\omega)$ is

$$S(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} s(t).e^{-i\omega t}\, dt,$$

where (.) denotes usual multiplication throughout the paper.

The inverse FT is defined as

$$s(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} S(\omega).e^{i\omega t}\, d\omega.$$

Let $s_1(t), s_2(t) \in L^2(\mathbb{R})$, then the convolution $s_1(t) * s_2(t)$ is defined as:

$$s_1(t) * s_2(t) = s_2(t) * s_1(t) = \int_{\mathbb{R}} s_1(t-\tau).s_2(\tau)\, d\tau = \int_{\mathbb{R}} s_1(\tau).s_2(t-\tau)\, d\tau.$$

As per convolution theorem in time domain:

$$S(s_1(t) * s_2(t)) = S(s_1(t)).S(s_2(t)),$$

whereas,

the dual of the convolution theorem says that

$$S(s_1(t).s_2(t)) = S(s_1(t)) * S(s_2(t)),$$

where $S(s_i(t))$ are the Fourier Transforms of the function $s_i(t)$, where $i = 1, 2$ [5].

Unlike conventional FT, FrFT can rotate the time frequency plane by any angle (α), not just 90 degrees. FrFT can provide meaningful representation of signals, especially those with time varying frequencies. In order to reduce the computational complexities, it is beneficial to look for fractional convolution operators and convolution theorems, since it has vital role in applications such as in filtering, harmonic analysis, pattern recognition and in allied areas of signal processing.

Fractional convolution extends the conventional signal combination process by introducing intermediate stages through FrFT. Previous works [3, 29] propose convolution operations incorporating additional chirp factors with signal product transformations. However, these approaches do not fully adhere to the classical Fourier transform's convolution theorem, as similar limitations are observed in [20, 21]. This paper aims to characterise the convolution operator to maintain consistency with the traditional Fourier convolution theorem. Moreover, we investigate how signal translations and convolutions behave within the time domain under varying FrFT angles (α). The proposed characterisation across multiple FrFT domains may offer a novel perspective on signal sampling and reconstruction [17]. Also, some of the properties of convolution operator alongwith graphical demonstrations are mentioned to validate the novelty.

2.2. **Fractional Fourier Transform.** The FrFT of a signal $s(t)$ at an angle α, denoted by $S_\alpha(u)$ [22], is expressed as follows:

$$S_\alpha(u) = \int_{\mathbb{R}} s(t).K_\alpha(u,t)\, dt,$$

$$s(t) = \int_{\mathbb{R}} S_\alpha(u).K_\alpha^*(u,t)\, du.$$

For integer m, the transform kernel is given by

$$K_\alpha(u,t) = \begin{cases} \delta(t-u), & \text{if } \alpha = 2m\pi \\ \delta(t+u), & \text{if } \alpha = (2m-1)\pi \\ \sqrt{\frac{1-i\cot\alpha}{2\pi}}\, e^{i\left(\frac{u^2+t^2}{2}\right)\cot\alpha - iut\csc\alpha}, & \text{if } \alpha \neq m\pi \end{cases} \tag{1}$$

where $K_\alpha^*(u,t)$ is a complex conjugate function of $K_\alpha(u,t)$.

3. Main Results

3.1. μ- generalised translation ($T_\mu(s(t)) = s(t\theta\mu)$). It is favourable if we have a time-domain operation that directly corresponds to a simple multiplication of their fractional Fourier transforms (FrFTs) in the FrFT domain, which results in the product $G_\alpha(u).S_\alpha(u)$. However, the time-domain expression for this multiplication, as outlined in [13][p.157], is not simple for direct computation, as the authors indicated. In this paper, by applying generalised convolution formula based on generalised translation operator, we efficiently interpret and execute this operation, validating our exploration of these concepts.

The authors probed a general signal(function) transform and its corresponding Fourier-like inverse [22], focusing on the mathematical properties and implications of this transformation.

$$s(t) = \int_{\mathbb{R}} \rho(u).S_\alpha(u).K(u,t)\,du, \tag{2}$$

$$S_\alpha(u) = \int_{\mathbb{R}} \rho(t).s(t).K^*(u,t)\,dt. \tag{3}$$

Further, the μ-generalised translation [31] of $s(t)$, i.e., $s(t\theta\mu)$ is mentioned as

$$s(t\theta\mu) = \int_{\mathbb{R}} \rho(u).S_\alpha(u).K^*(u,t).K(u,\mu)\,du. \tag{4}$$

The parameter θ in the function $s(t\theta\mu)$ represents the generalised time delay operator corresponding to generalised translation. Here, $K(u,\mu)$ is the kernel of transformation, where 't' is replaced by the translation parameter 'μ', $K^*(u,t)$ represents the conjugate of kernel of the transformation and $\rho(u)$ is the weight function. The shift property of the conventional FT, given by $F(s(t-\mu)) = S(u).e^{-iu\mu}$, syncs with the defined generalised translation.

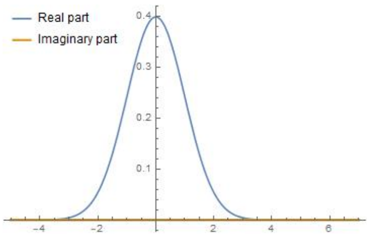
Considering the specific case of FrFT, where the weight function is constant and the kernel matching the FrFT, equation 4 simplifies to

$$s(t\theta\mu) = \frac{|\csc\alpha|}{\sqrt{2\pi}}.e^{i\left(\frac{\mu^2-t^2}{2}\right)\cot\alpha}.\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} S_\alpha(u).e^{iu(t-\mu)\csc\alpha}\,du. \tag{5}$$

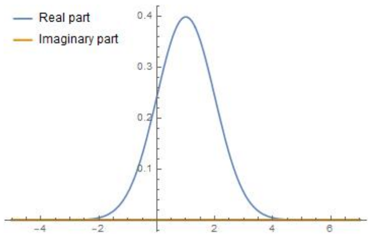
We can derive equation 5 as follows.

$$\begin{aligned}
s(t\theta\mu) &= \sqrt{\frac{1+i\cot\alpha}{2\pi}}.\sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_{\mathbb{R}} S_\alpha(u).e^{i\left(\frac{u^2+\mu^2}{2}\right)\cot\alpha+iu\mu\csc\alpha} \\
&\quad\times e^{-i\left(\frac{u^2+t^2}{2}\right)\cot\alpha-iut\csc\alpha}\,du \\
&= \frac{|\csc\alpha|}{2\pi}.e^{i\left(\frac{\mu^2-t^2}{2}\right)\cot\alpha}\int_{\mathbb{R}} S_\alpha(u).e^{iu(t-\mu)\csc\alpha}\,du \\
&= \frac{|\csc\alpha|}{\sqrt{2\pi}}.e^{i\left(\frac{\mu^2-t^2}{2}\right)\cot\alpha}.\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} S_\alpha(u).e^{iu(t-\mu)\csc\alpha}\,du.
\end{aligned}$$

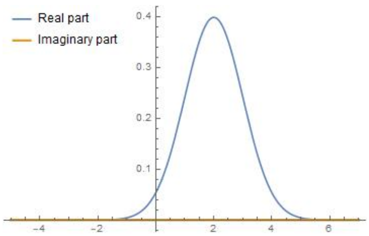
For $\alpha = \frac{(2m+1)\pi}{2}$, $m \in \mathbb{Z}$, equation 5 can be reduced to $s(t-\mu)$, which is the normal translation along time domain. As an example, from figure 1, normal translation



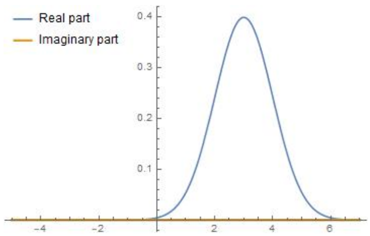
(A) $\mu = 0$.



(B) $\mu = 1$.

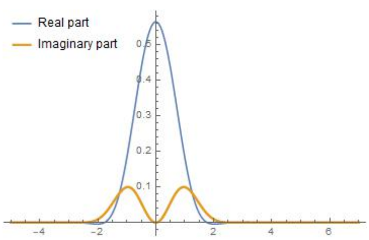


(C) $\mu = 2$.

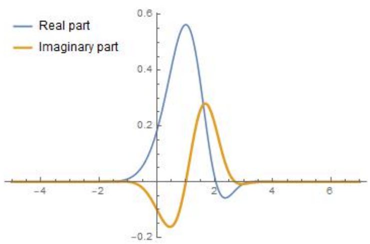


(D) $\mu = 3$.

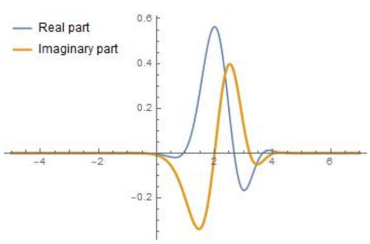
FIGURE 1. Translation of Gaussian function computed at various translations for $\alpha = \frac{\pi}{2}$



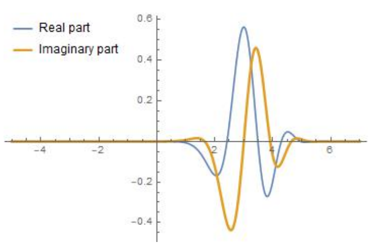
(A) $\mu = 0$.



(B) $\mu = 1$.

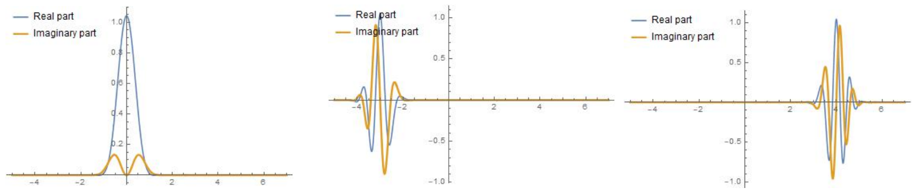


(C) $\mu = 2$.



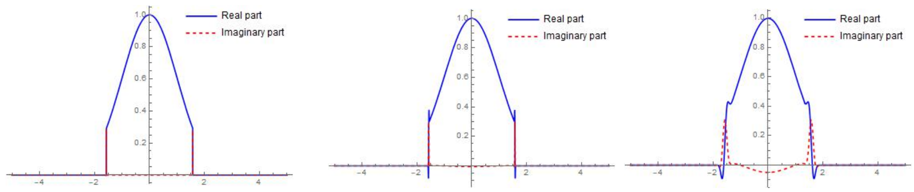
(D) $\mu = 3$.

FIGURE 2. Translation of Gaussian function computed at various translations for $\alpha = \frac{\pi}{4}$

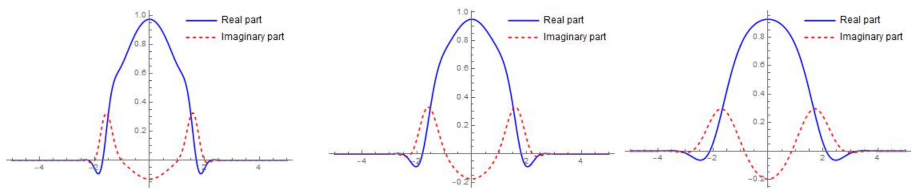


(A) $\mu = 0$. (B) $\mu = -3$. (C) $\mu = 4$.

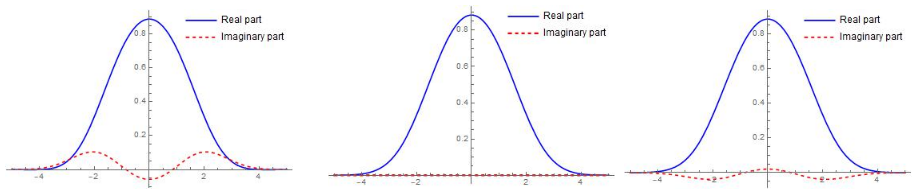
FIGURE 3. Translation of Gaussian function computed at various translations for $\alpha = \frac{\pi}{8}$



(A) α=0.001 (B) α= 0.01 (C) α= 0.1



(D) $\alpha = 0.29$. (E) $\alpha = 0.44$. (F) $\alpha = 0.76$.



(G) $\alpha = 1.35$. (H) $\alpha = \frac{\pi}{2}$. (I) $\alpha = 1.65$.

FIGURE 4. Convolution of box function and Gaussian function computed for various values of α.

can be obtained when we fix $\alpha = \frac{\pi}{2}$. The distinction between conventional FT and FrFT is obvious with restricted fractional Fourier kernel's influence, manifesting in transformations under generalised translation. Varying translation parameters yield notable changes when the order (α) changes, highlighting its pivotal role in the transformation process. These changes can be seen from figure 2 and figure 3.

Using the μ-generalised translation of the signal $s(t)$ by μ, a generalised convolution in the time domain is derived, which is defined as

$$
\begin{aligned}
s(t)\Theta q(t) &= \int_{\mathbb{R}} s(\mu).q(t\theta\mu)\,d\mu \\
&= \frac{|\csc\alpha|}{\sqrt{2\pi}}\int_{\mathbb{R}} s(\mu).e^{i\left(\frac{\mu^2-t^2}{2}\right)\cot\alpha}\int_{\mathbb{R}} Q_\alpha(u).e^{iu(t-\mu)\csc\alpha}\,du\,d\mu \\
&= \frac{|\csc\alpha|\sqrt{1-i\cot\alpha}}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(\mu).g(t') \\
&\times e^{i\left(\frac{\mu^2-t^2+u^2+t'^2}{2}\right)\cot\alpha}.e^{-iu(t'+t-\mu)\csc\alpha}\,du\,d\mu\,dt',
\end{aligned}
\tag{6}
$$

where Θ denotes generalised convolution in time domain [18, 22].

The introduction of FrFT paved the way for more general concept of fractional convolution with the building blocks of the traditional convolution operators, expected to be more versatile and enhanced flexibility.

Noticeably, the FT and FrFT exhibit distinct differences. Upon applying generalised convolution, the impact of the fractional Fourier kernel becomes evident through the transformation across various fractional values of the order (α). The visual representations reveals a pronounced effect when box function is convoluted with Gaussian function, see figure 4.

3.2. **Generalised Convolution Theorem (GCT) of FrFT in Time Domain.**

Theorem 3.1. *Let $S^1_\alpha(u)$ and $S^2_\alpha(u)$ are the fractional Fourier transforms of $s_1(t)$ and $s_2(t)$, respectively. The fractional Fourier transform of generalised convolution of $s_1(t)$ and $s_2(t)$ is equal to the product of fractional Fourier transforms of $s_1(t)$ and $s_2(t)$ in the transformed domain, i.e,*

$$
S_\alpha(s_1(t)\Theta s_2(t)) = S_\alpha(s_1(t)).S_\alpha(s_2(t)) = S^1_\alpha(u).S^2_\alpha(u) \tag{7}
$$

Proof :
Let $S_{-\alpha}(.)$ denotes the inverse FrFT. Then,

$$
\begin{aligned}
S_{-\alpha}(S^1_\alpha(u).S^2_\alpha(u)) &= \int_{\mathbb{R}} S^1_\alpha(u).S^2_\alpha(u).\sqrt{\frac{1+i\cot\alpha}{2\pi}}.e^{-i\left(\frac{u^2+t^2}{2}\right)\cot\alpha+iut\csc\alpha}\,du \\
&= \frac{|\csc\alpha|}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} s_1(\mu).e^{i\left(\frac{u^2+\mu^2}{2}\right)\cot\alpha-iu\mu\csc\alpha}\,d\mu \\
&\times S^2_\alpha(u).e^{-i\left(\frac{u^2+t^2}{2}\right)\cot\alpha+iut\csc\alpha}\,du \\
&= \frac{|\csc\alpha|}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} s_1(\mu).S^2_\alpha(u).e^{-i\left(\frac{t^2-\mu^2}{2}\right)\cot\alpha+iu(t-\mu)\csc\alpha}\,du\,d\mu \\
&= s_1(t)\Theta s_2(t).
\end{aligned}
$$

This equation is crucial for analog filtering [3] in the FrFT domain. Notably, when $\alpha = \frac{\pi}{2}$, the generalised convolution becomes conventional Fourier transform convolution as evident from equation 7.

3.3. Some Properties of Generalised Convolution Operator.

proposition 3.1. *Suppose $s_1(t)$, $s_2(t)$ and $s_3(t)$ are three signals with fractional Fourier transforms $S^1_\alpha(u)$, $S^2_\alpha(u)$ and $S^3_\alpha(u)$, respectively. For $c, c_1, c_2 \in \mathbb{K}(\mathbb{R}$ or $\mathbb{C})$, the following properties holds*

(i) $s_1(t)\Theta s_2(t) = s_2(t)\Theta s_1(t)$.
(ii) $c.s_1(t)\Theta s_2(t) = s_1(t)\Theta c.s_2(t) = c.(s_1(t)\Theta s_2(t))$.
(iii) $s_1(t)\Theta(s_2(t) + s_3(t)) = (s_1(t)\Theta s_2(t)) + (s_1(t)\Theta s_3(t))$.
(iv) $(c_1.s_1(t) + c_2.s_2(t))\Theta s_3(t) = c_1(s_1(t)\Theta s_3(t)) + c_2(s_2(t)\Theta s_3(t))$.

Proof:

(i)

$$\begin{aligned}
s_2(t)\Theta s_1(t) &= \int_{\mathbb{R}} s_2(\mu).s_1(t\theta\mu)d\mu \\
&= \int_{\mathbb{R}} \frac{|csc|}{2\pi}.s_2(\mu).e^{i\left(\frac{\mu^2-t^2}{2}\right)cot\alpha} \int_{\mathbb{R}} S^1_\alpha(u).e^{iu(t-\mu)csc\alpha}dud\mu \\
&= \frac{|csc|}{2\pi}.\sqrt{\frac{1-j\cot\alpha}{2\pi}} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} s_2(\mu).e^{i\left(\frac{\mu^2-t^2}{2}\right)cot\alpha}.s_1(t') \\
&\times e^{i\left(\frac{u^2+t'^2}{2}\right)cot\alpha - iut'csc\alpha}.e^{iu(t-\mu)csc\alpha}dud\mu dt' \\
&= \frac{|csc\alpha|}{2\pi}.\sqrt{\frac{1-jcot\alpha}{2\pi}} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} s_2(t').s_1(\mu).e^{i\left(\frac{t'^2-t^2+u^2+\mu^2}{2}\right)cot\alpha} \\
&\times e^{-iu(t'+\mu-t)csc\alpha}dud\mu dt' \\
&= \int_{\mathbb{R}} s_1(\mu).s_2(t\theta\mu)d\mu \\
&= s_1(t)\Theta s_2(t).
\end{aligned}$$

(ii)

$$\begin{aligned}
c.s_1(t)\Theta s_2(t) &= \int_{\mathbb{R}} c.s_1(t).s_2(t\theta\mu)d\mu \\
&= \int_{\mathbb{R}} c.s_1(\mu).\frac{|csc|}{2\pi}.e^{i\left(\frac{\mu^2-t^2}{2}\right)cot\alpha} \int_{\mathbb{R}} S^2_\alpha(u).e^{iu(t-\mu)csc\alpha}du. \\
&= \int_{\mathbb{R}} c.s_2(\mu).\frac{|csc|}{2\pi}.e^{i\left(\frac{\mu^2-t^2}{2}\right)cot\alpha} \int_{\mathbb{R}} S^1_\alpha(u).e^{iu(t-\mu)csc\alpha}du. \\
&= s_2(t)\Theta c.s_1(t) = c(s_2(t)\Theta s_1(t)).
\end{aligned}$$

(iii)

$$
\begin{aligned}
s_1(t)\Theta(s_2(t)+s_3(t)) &= \int_{\mathbb{R}} s_1(\mu)(s_2+s_3)(t\theta\mu)d\mu. \\
&= \int_{\mathbb{R}} s_1(\mu).\frac{|csc\alpha|}{2\pi}.e^{i\left(\frac{\mu^2-t^2}{2}\right)cot\alpha}\int_{\mathbb{R}}(S_\alpha^2+S_\alpha^3)(u) \\
&\times e^{i(t-\mu)csc\alpha}du \\
&= \int_{\mathbb{R}} s_1(\mu).\frac{|csc\alpha|}{2\pi}.e^{i\left(\frac{\mu^2-t^2}{2}\right)cot\alpha}\int_{\mathbb{R}} S_\alpha^2(u).e^{i(t-\mu)csc\alpha}du \\
&+ \int_{\mathbb{R}} s_1(\mu).\frac{|csc\alpha|}{2\pi}.e^{i\left(\frac{\mu^2-t^2}{2}\right)cot\alpha}\int_{\mathbb{R}} S_\alpha^3(u).e^{i(t-\mu)csc\alpha}du \\
&= (s_1(t)\Theta s_2(t)) + (s_1(t)\Theta s_3(t)).
\end{aligned}
$$

(iv) Using property (ii) and (iii),

$$
\begin{aligned}
(c_1.s_1(t)+c_2.s_2(t))\Theta s_3(t) &= (c_1.s_1(t)\Theta s_3(t)) + (c_2.s_2(t)\Theta s_3(t)) \\
&= c_1(s_1(t)\Theta s_3(t)) + c_2(s_2(t)\Theta s_3(t)).
\end{aligned}
$$

Hence the proposition.

Remark: If we consider the case where, $s_1(t)=s_2(t)=e^{\frac{-t^2}{2}}$ and $s_3(t)=e^{\left(\frac{-t^2}{2}+2t\right)}$ for $\alpha=\frac{\pi}{6}$, associative property does not holds.

4. Conclusion

In this paper, we developed a characterisation of convolution operator in time domain using generalised translation of FrFT. The results are verified with suitable examples. When $\alpha=\frac{(2m+1)\pi}{2}$, $m\in\mathbb{Z}$, only, the translation retains the characteristics of the conventional FT. However, when $\alpha\neq\frac{m\pi}{2}, m\in\mathbb{Z}$, the translations of the function shows the supriority of FrFT over FT. Using some examples, we have shown that the generalised fractional convolution operator preserves the convolution properties of conventional FT. Moreover, we see that under certain restrictions, some properties of conventional FT convolution operator shaking hands with our proposed fractional Fourier convolution operator. In near future, we are planning to explore further applications of FrFT convolution methods in image processing.

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